Generating Functions

1. Probability-Generating Function:

Let x be a r.v. (usually discrete). Then, its PGF is $r_x(t) = E(t^x)$, $\forall t \in \mathbb{R}$.

E.g. 1 $x \sim Bin(n, \theta)$

Find $r_x(t)$.

$$r_x(t) = E(t^x)$$
$$= \sum_{x=0}^{n} t^x p_x(x)$$
$$= \sum_{x=0}^{n} t^x \binom{n}{x} \theta^x (1-\theta)^{n-x} \leftarrow \text{Binominal Expansion}$$
$$= (1-\theta+t\theta)^n$$

Note: $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$ is the binominal expansion.

E.g. 2 $x \sim Geo(\theta)$

Find $r_x(t)$

$$r_x(t) = E(t^x)$$
$$= \sum_{x=0} t^x p_x(x)$$
$$= \sum_{x=0} t^x (1-\theta)^x \theta$$
$$= \theta \underbrace{\sum (t-t\theta)^x}_{\text{Geometric Sequence}}$$

$$= \theta\left(\frac{a}{1-r}\right)$$

$a = (t - t\theta)^0 = 1$

$r = (t - t\theta)$

$$= \frac{\theta}{1-(t-t\theta)}$$

$$= \frac{\theta}{1-(1-\theta)t}$$

List of PGF For Discrete Distribution:

1. $X \sim Bi(n, \theta) \rightarrow r_X^{(t)} = (1-\theta+t\theta)^n$
2. $X \sim Geo(\theta) \rightarrow r_X^{(t)} = \dfrac{\theta}{1-(1-\theta)t}$
3. $X \sim Po(\lambda) \longrightarrow r_X^{(t)} = e^{\lambda(t-1)}$
4. $X \sim Neg\text{-}Bi(r, \theta) \longrightarrow r_X^{(t)} = \theta^r(1-t(1-\theta))^{-r}$

Thm:

Let X be a discrete r.v. whose possible values are all non-negative. Assume that $r_X^{(t_0)} < \infty$ for some $t_0 > 0$. Then, $r^{(k)}{}_X(0) = k!\, P(X=k)$. $r^{(k)}{}_X{}^{(0)}$ is the k^{th} derivative of x.

2. Moment Generating Function:

Let X be any r.v. Then, its MGF is defined by $m_X(t) = E(e^{tX})$.

Note:

1. $r_X(t) = E(t^X)$
$= E(e^{\ln(t)X})$
$= m_X(\ln t)$

2. $m_X(t) = E(e^{tX})$
$= E((e^t)^X)$
$= r_X(e^t)$

List of MGF For Continuous Distribution:

1. $X \sim Exp(\lambda) \rightarrow m_X(t) = \frac{\lambda}{\lambda - t}$

2. $X \sim N(\mu, \sigma^2) \rightarrow m_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$
$X \sim N(0,1) \rightarrow m_X(t) = e^{\frac{1}{2}t^2}$

Thms:

1. Let $Y = a + bX$. Then, $m_Y(t) = e^{at} \cdot m_X(bt)$

Proof:

$m_Y(t) = E(e^{tY})$
$= E(e^{t(a+bX)})$
$= E(e^{at} \cdot e^{(bX)t})$
$= e^{at} \cdot E(e^{(bt)X})$
$= e^{at} \cdot m_X(bt)$, as wanted

2. Let X be any r.v. Suppose that for some $t_0 > 0$ that $m_X(t) < \infty$ when $t \in (-t_0, t_0)$. Then, $m^{(k)}_X(0) = E(X^k)$ where $m^{(k)}_X$ is the k^{th} derivative of X.

3. If $X_1, X_2, \ldots X_n$ are independent, then
   1. $r_{X_1+X_2+\ldots X_n}(t) = r_{X_1}(t) \cdot r_{X_2}(t) \cdot \ldots r_{X_n}(t)$
   2. $m_{X_1+X_2+\ldots X_n}(t) = m_{X_1}(t) \cdot m_{X_2}(t) \cdot \ldots m_{X_n}(t)$